第五次作业参考答案

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习题 1

在库伦规范下，矢量场的模式展开写为（相对论性归一化）

$$\vec{A}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\sum_{\lambda}\left[\vec{\epsilon}_{\lambda}{}^{*}(\vec{k})a_{\lambda}(\vec{k})e^{-ik\cdot x}+\vec{\epsilon}_{\lambda}(\vec{k})a_{\lambda}^{\dagger}(\vec{k})e^{ik\cdot x}\right] \tag{1}$$

其中 λ 是极化指标，求和仅对两个物理极化进行。从场的等时正则对易关系

$$\left[A_i(\vec{x}),\pi^j(\vec{y})\right]=\left[A_i(\vec{x}),E^j(\vec{y})\right]=i\left(\delta_i^j-\frac{\partial_i\partial^j}{\vec{\nabla}^2}\right)\delta^3(\vec{x}-\vec{y}) \tag{2}$$

计算产生湮灭算符的对易关系。

解:

由矢量场算符的模式

$$\vec{A}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\sum_{\lambda}\left[\vec{\epsilon}_{\lambda}{}^{*}(\vec{k})a_{\lambda}(\vec{k})e^{-ik\cdot x}+\vec{\epsilon}_{\lambda}(\vec{k})a_{\lambda}^{\dagger}(\vec{k})e^{ik\cdot x}\right] \tag{3}$$

其中 $\omega_{\vec{k}}=|\vec{k}|$，得到

$$\partial_0\vec{A}(x)=\int\frac{d^3k}{(2\pi)^3 2w_k}\left(-i\omega_{\vec{k}}\right)\sum_{\lambda}\left[\vec{\epsilon}_{\lambda}{}^{*}(\vec{k})a_{\lambda}(\vec{k})e^{-ik\cdot x}-\vec{\epsilon}_{\lambda}a_{\lambda}^{\dagger}(\vec{k})e^{ik\cdot x}\right]. \tag{4}$$

那么，

$$\begin{aligned}
&\int d^3x e^{i\vec{p}\cdot\vec{x}}\vec{A}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\sum_{\lambda}\int d^3x e^{i\vec{p}\cdot\vec{x}}\left[\vec{\epsilon}_{\lambda}{}^{*}(\vec{k})a_{\lambda}(\vec{k})e^{-ik\cdot x}+\vec{\epsilon}_{\lambda}a_{\lambda}^{\dagger}(k)e^{ik\cdot x}\right]\\
&=\int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\sum_{\lambda}\left[\vec{\epsilon}_{\lambda}{}^{*}(\vec{k})a_{\lambda}(\vec{k})e^{-i\omega_{\vec{k}}t}\cdot(2\pi)^3\delta^{(3)}(\vec{p}+\vec{k})+\vec{\epsilon}_{\lambda}a_{\lambda}^{\dagger}(\vec{k})e^{i\omega_{\vec{k}}t}\cdot(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})\right]\\
&=\frac{1}{2\omega_{\vec{p}}}\sum_{\lambda}\left[\vec{\epsilon}_{\lambda}{}^{*}(-\vec{p})a_{\lambda}(-\vec{p})e^{-i\omega_{\vec{p}}t}+\vec{\epsilon}_{\lambda}(\vec{p})a^{\dagger}(\vec{p})e^{i\omega_{\vec{p}}t}\right]
\end{aligned} \tag{5}$$

则

$$\begin{aligned}
\vec{\epsilon}_{\lambda}{}^{*}(\vec{p})\cdot\int d^3x e^{i\vec{p}\cdot\vec{x}}\vec{A}(x)&=\frac{1}{2\omega_{\vec{p}}}\sum_{r}\vec{\epsilon}_{\lambda}{}^{*}(\vec{p})\cdot\vec{\epsilon}_{r}^{*}(\vec{p})a_{x}(-\vec{p})e^{-i\omega_{\vec{p}}t}+\frac{1}{2\omega_{p}}a_{\lambda}^{\dagger}(\vec{p})e^{i\omega_{p}t}\\
\Rightarrow\int d^3x e^{-ip\cdot x}\vec{\epsilon}_{\lambda}{}^{*}(\vec{p})\vec{A}(x)&=\frac{1}{2\omega_{\vec{p}}}e^{-2i\omega_{\vec{p}}t}\sum_{r}\vec{\epsilon}_{\lambda}{}^{*}\cdot\vec{\epsilon}_{r}{}^{*}(-\vec{p})a_{r}(-\vec{p})+\frac{1}{2\omega_{\vec{p}}}a_{\lambda}^{\dagger}(\vec{p})
\end{aligned} \tag{6}$$

同理，

$$\int d^3x e^{-ip\cdot x}\vec{\epsilon}_\lambda^{\,*}(\vec{p})\partial_0\vec{A}(x) = \frac{-i}{2}e^{-i\omega_{\vec{p}}t}\vec{\epsilon}_\lambda^{*}\cdot\sum_r\vec{\epsilon}_r^{*}(-\vec{p})a_r(-\vec{p}) + \frac{i}{2}a_\lambda^\dagger(\vec{p}). \tag{7}$$

所以产生算符可以由场算符表示为，

$$\begin{aligned} a_\lambda^\dagger(\vec{p}) &= \int d^3x e^{-ip\cdot x}\vec{\epsilon}_\lambda^{\,*}(\vec{p})\cdot\left(\omega_{\vec{p}}\vec{A}(x) - i\partial_0\vec{A}(x)\right) \\ &= -i\int d^3x e^{-ip\cdot x}\vec{\epsilon}_\lambda^{\,*}(\vec{p})\cdot\left(i\omega_{\vec{p}} + \partial_0\right)\vec{A}(x) \\ &= -i\int d^3x e^{-ip\cdot x}\vec{\epsilon}_\lambda^{\,*}(\vec{p})\cdot\overleftrightarrow{\partial}_0\vec{A}(x). \end{aligned} \tag{8}$$

同理，

$$a_\lambda(\vec{p}) = i\int d^3x e^{ip\cdot x}\vec{\epsilon}_\lambda(\vec{p})\cdot\overleftrightarrow{\partial}_0\vec{A}(x), \tag{9}$$

则初始时刻的产生湮灭算符为

$$\begin{aligned} a_\lambda^\dagger(\vec{p})_{t=0} &= e^{-iHt}a^\dagger(\vec{p})e^{iHt} = a^\dagger(\vec{p})e^{i\omega_{\vec{p}}t} \\ &= -i\int d^3x e^{i\vec{p}\cdot\vec{x}}\vec{\epsilon}_\lambda^{\,*}(\vec{p})\cdot\left(\partial_0 + i\omega_{\vec{p}}\right)\vec{A}(\vec{x}) \end{aligned} \tag{10}$$

和

$$a_\lambda(\vec{p})_{t=0} = a(\vec{p})e^{-i\omega_{\vec{p}}t} = i\int d^3x e^{-i\vec{p}\cdot\vec{x}}\vec{\epsilon}_\lambda\cdot\left(\partial_0 - i\omega_p\right)\vec{A}(\vec{x}) \tag{11}$$

注意到对于自由场 $E^i(x) = -\partial_0 A^i(x)$。于是等时对易子可以表示为

$$\begin{aligned} &\left[a_\lambda(\vec{k}), a_{\lambda'}^\dagger(\vec{k}')\right]_{t=t_1} = \left[a_\lambda(\vec{k})_{t=0}, a_{\lambda'}(\vec{k}')_{t=0}\right] \\ =&\left[i\int d^3x e^{-ik\cdot\vec{x}}\vec{\epsilon}_\lambda\cdot\left(\partial_0 - i\omega_k\right)\vec{A}(\vec{x}), -i\int d^3x' e^{i\vec{k}'\cdot\vec{x}'}\vec{\epsilon}_{\lambda'}^{\,*}\cdot\left(\partial_0 + i\omega_{\vec{k}'}\right)\vec{A}\left(\vec{x}'\right)\right] \\ =&\int d^3x\int d^3x' e^{-i\vec{k}\cdot\vec{x}+i\vec{k}'\cdot\vec{x}'}\left[\vec{\epsilon}_\lambda\cdot\left(-\vec{E}(\vec{x}) - i\omega_{\vec{k}}\vec{A}(\vec{x})\right), \vec{\epsilon}_{\lambda'}^{\,*}\cdot\left(-\vec{E}\left(\vec{x}'\right) + i\omega_{\vec{k}'}\cdot\vec{A}(\vec{x}')\right)\right] \\ =&-\int d^3x\int d^3x' e^{-i\vec{k}\vec{x}+\vec{x}'\vec{k}'\vec{x}'}\left(i\omega_{\vec{k}'}\epsilon_{\lambda'}^{i\,*}\epsilon_{\lambda,j}\left[A_i\left(\vec{x}'\right), E^j(\vec{x})\right] + i\omega_{\vec{k}}\epsilon_\lambda^i\epsilon_{\lambda',j}^*\left[A_i(\vec{x}), E^j\left(\vec{x}'\right)\right]\right) \\ =&-\int d^3x\int d^3x' e^{-i\vec{k}\cdot\vec{x}+i\vec{k}'\cdot\vec{x}'}\left(i\omega_{\vec{k}'}\epsilon_{\lambda'}^{i*}(\vec{k}')\epsilon_{\lambda,j}(\vec{k})iP_i^j\delta^{(3)}(\vec{x}-\vec{x}) + i\omega_{\vec{k}}\epsilon_\lambda^i(\vec{k})\epsilon_{\lambda',j}^*\left(\vec{k}'\right)iP_i^j\delta^{(3)}\left(\vec{x}-\vec{x}'\right)\right) \\ =&-\int d^3x e^{i\left(\vec{k}'-\vec{k}\right)\cdot\vec{x}}\left(i\omega_{\vec{k}'}\epsilon_{\lambda'}^{i*}(\vec{k}')\epsilon_{\lambda,j}(\vec{k})iP_i^j + i\omega_{\vec{k}}\epsilon_\lambda^i(\vec{k})\epsilon_{\lambda',j}^*\left(\vec{k}'\right)iP_i^j\right) \\ =&(2\pi)^3\delta^{(3)}\left(\vec{k}-\vec{k}'\right)\omega_{\vec{k}}P_i^j(\vec{k})\left(\epsilon_{\lambda'}^{i*}(\vec{k})\epsilon_{\lambda,j}(\vec{k}) + \epsilon_{\lambda',j}^*(\vec{k})\epsilon_\lambda^i(\vec{k})\right) \end{aligned} \tag{12}$$

注意到 $P_i^j(\vec{k}) = \sum_{\lambda=\pm}\vec{\epsilon}_{\lambda,i}^{\,*}(\vec{k})\vec{\epsilon}_\lambda^{\,j}(\vec{k}) = \sum_{\lambda=\pm}\vec{\epsilon}_\lambda^{\,j*}(\vec{k})\vec{\epsilon}_{\lambda,i}(\vec{k})$ 则

$$P_i^j(\vec{k})\epsilon_{\lambda'}^{i*}(\vec{k})\epsilon_{\lambda,j}(\vec{k}) = P_i^j(\vec{k})\epsilon_{\lambda',j}^*(\vec{k})\epsilon_\lambda^i(\vec{k}) = \delta_{\lambda\lambda'} \tag{13}$$

综上有

$$\left[a_\lambda(\vec{k}), a_{\lambda'}^\dagger(\vec{k}')\right]_{t=t_1} = \delta_{\lambda\lambda'}2\omega_{\vec{k}}(2\pi)^3\delta^{(3)}\left(\vec{k}-\vec{k}'\right) \tag{14}$$

习题 2

设 $U(\omega)=\exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)$ 定义了一个无穷小洛伦兹变换，其中 $\omega_{\mu\nu}=-\omega_{\nu\mu}$ 是反对称张量。按照矢量场在洛伦兹变换下的定义式：

$$U^{-1}(\omega)A^\rho(x)U(\omega)=\Lambda(\omega)^\rho_\sigma A^\sigma\left(\Lambda^{-1}(\omega)x\right) \tag{15}$$

其中 $\Lambda(\omega)^\rho_\sigma=\delta^\rho_\sigma+\omega^\rho_\sigma$ ，求

$$[M^{\mu\nu},A^\rho(x)] \tag{16}$$

解：

对于无穷小变化，左边为

$$\begin{aligned} U^{-1}(\omega)A^\rho(x)U(\omega) &\approx A^\rho(x)-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}A^\rho(x)+\frac{i}{2}A^\rho(x)\omega_{\mu\nu}M^{\mu\nu} \\ &= A^\rho(x)-\frac{i}{2}\omega_{\mu\nu}\left[M^{\mu\nu},A^\rho(x)\right] \end{aligned} \tag{17}$$

右边为

$$\begin{aligned} \Lambda(\omega)^\rho_\sigma A^\sigma\left(\Lambda^{-1}(\omega)x\right) &\approx \left(\delta^\rho_\sigma+\omega^\rho_\sigma\right)\left(A^\sigma(x)-\omega^\mu_\nu x^\nu\partial_\mu A^\sigma(x)\right) \\ &\approx A^\rho(x)+\omega^{\rho\sigma}A_\sigma(x)-\omega^{\mu\nu}x_\nu\partial_\mu A^\rho(x) \\ &= A^\rho(x)+\omega_{\mu\nu}\left(-x^\nu\partial^\mu A^\rho(x)+\eta^{\mu\rho}A^\nu(x)\right) \\ &= A^\rho(x)+\frac{1}{2}\omega_{\mu\nu}\left((x^\mu\partial^\nu-x^\nu\partial^\mu)A^\rho(x)+(\eta^{\mu\rho}\delta^\nu_\sigma-\eta^{\nu\rho}\delta^\mu_\sigma)A^\sigma(x)\right) \end{aligned} \tag{18}$$

因此，

$$[M^{\mu\nu},A^\rho(x)]=i\left((x^\mu\partial^\nu-x^\nu\partial^\mu)A^\rho(x)+(\eta^{\mu\rho}\delta^\nu_\sigma-\eta^{\nu\rho}\delta^\mu_\sigma)A^\sigma(x)\right). \tag{19}$$

记 $(\mathcal{J}^{\mu\nu})^\alpha{}_\beta=i(\eta^{\mu\alpha}\delta^\nu_\beta-\eta^{\nu\alpha}\delta^\mu_\beta)$ 与 $J^{\mu\nu}=i(x^\mu\partial^\nu-x^\nu\partial^\mu)$, 即洛伦兹群的两种表示。则上式可以写为

$$[M^{\mu\nu},A^\rho(x)]=J^{\mu\nu}A^\rho(x)+(\mathcal{J}^{\mu\nu})^\rho{}_\sigma A^\sigma(x) \tag{20}$$

习题 3

接上题，对 $A^\rho(x)$ 在洛伦兹规范下作模式展开。计算 $\left[M^{\mu\nu},a(\vec{k},\lambda)\right]$ 和 $\left[M^{\mu\nu},a^\dagger(\vec{k},\lambda)\right]$, 其中 $\lambda=0,1,2,3$ 是极化指标。

解：

场方程 $\partial^2A^\mu=0$ 的解可以模式展开为

$$A^\mu(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\sum_{\lambda=0}^{3}\left[\epsilon^{*\mu}(\vec{k},\lambda)a(\vec{k},\lambda)e^{-ikx}+\epsilon^\mu(\vec{k},\lambda)a^\dagger(\vec{k},\lambda)e^{ikx}\right]\Bigg|_{k^0=\omega_{\vec{k}}} \tag{21}$$

因为

$$
\begin{aligned}
&\int d^3xe^{-i\vec{p}\cdot\vec{x}}A^\mu(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\sum_\lambda\int d^3xe^{-i\vec{p}\cdot\vec{x}}\left[\epsilon^{*\mu}(\vec{k},\lambda)a(\vec{k},\lambda)e^{-ikx}+\epsilon^\mu(\vec{k},\lambda)a^\dagger(\vec{k},\lambda)e^{ikx}\right]\\
&=\frac{1}{2\omega_{\vec{p}}}\sum_\lambda\left[\epsilon^{*\mu}(\vec{k},\lambda)a(\vec{k},\lambda)e^{-i\omega_{\vec{p}}t}+\epsilon^\mu(-\vec{k},\lambda)a^\dagger(-\vec{k},\lambda)e^{i\omega_{\vec{p}}t}\right]
\end{aligned}
\tag{22}
$$

即

$$
\int d^3xe^{ip\cdot x}A^\mu(x)=\frac{1}{2\omega_{\vec{p}}}\sum_\lambda\left[\epsilon^{*\mu}(\vec{k},\lambda)a(\vec{k},\lambda)+\epsilon^\mu(-\vec{k},\lambda)a^\dagger(-\vec{k},\lambda)e^{2i\omega_{\vec{p}}t}\right]
\tag{23}
$$

则易求，

$$
\begin{aligned}
a^\dagger(\vec{p},\lambda)&=-i\int d^3xe^{-ip\cdot x}\epsilon^{\mu*}(\vec{p},\lambda)\overleftrightarrow{\partial}_0A_\mu(x)\\
a(\vec{p},\lambda)&=i\int d^3xe^{ip\cdot x}\epsilon^{\mu}(\vec{p},\lambda)\overleftrightarrow{\partial}_0A_\mu(x)
\end{aligned}
\tag{24}
$$

于是，

$$
\begin{aligned}
&\left[M^{\mu\nu},a(\vec{k},\lambda)\right]\\
=&\left[M^{\mu\nu},i\int d^3xe^{ik\cdot x}\epsilon^\rho(\vec{k},\lambda)(\partial_0-i\omega_{\vec{k}})A_\rho(x)\right]\\
=&i\int d^3xe^{ik\cdot x}\epsilon_\rho(\vec{k},\lambda)\left[M^{\mu\nu},(\partial_0-i\omega_{\vec{k}})A^{+\rho}(x)\right]+i\int d^3xe^{ik\cdot x}\epsilon_\rho(\vec{k},\lambda)\left[M^{\mu\nu},(\partial_0-i\omega_{\vec{k}})A^{-\ \rho}(x)\right]
\end{aligned}
\tag{25}
$$

其中，

$$
A^{+\mu}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\sum_{\lambda=0}^{3}\epsilon^{*\mu}(\vec{k},\lambda)a(\vec{k},\lambda)e^{-ikx}
\tag{26}
$$

$$
A^{-\mu}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\sum_{\lambda=0}^{3}\epsilon^{\mu}(\vec{k},\lambda)a^\dagger(\vec{k},\lambda)e^{ikx}
\tag{27}
$$

不妨记 $\alpha^\mu(\vec{k})=\sum_\lambda\epsilon^{*\mu}(\vec{k},\lambda)a(\vec{k},\lambda)$. 那么第一项中

$$
\begin{aligned}
&\int d^3xe^{ik\cdot x}\left[M^{ij},A^{+\rho}(x)\right]\\
=&\int d^3xe^{ik\cdot x}\left(J^{ij}A^{+\rho}(x)+\left(\mathcal{J}^{ij}\right)^\rho{}_\sigma A^{+\sigma}(x)\right)\\
=&\int d^3xe^{ik\cdot x}\left(J^{ij}\delta^\rho_\sigma+\left(\mathcal{J}^{ij}\right)^\rho{}_\sigma\right)\int\frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}\alpha^\sigma(\vec{p})e^{-ipx}\\
=&\int\frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}\alpha^\sigma(\vec{p})\left(\tilde{J}^{ji}\delta^\rho_\sigma+\left(\mathcal{J}^{ij}\right)^\rho{}_\sigma\right)\int d^3xe^{i(k-p)x}\\
=&\int\frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}\alpha^\sigma(\vec{p})\left(-\tilde{J}^{ij}\delta^\rho_\sigma+\left(\mathcal{J}^{ij}\right)^\rho{}_\sigma\right)(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})x^0}\\
=&\int\frac{d^3p}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})\frac{1}{2\omega_{\vec{p}}}e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})x^0}\left(\tilde{J}^{ij}\delta^\rho_\sigma+\left(\mathcal{J}^{ij}\right)^\rho{}_\sigma\right)\alpha^\sigma(\vec{p})\\
=&\frac{1}{2\omega_{\vec{k}}}\left(\tilde{J}^{ij}\delta^\rho_\sigma+\left(\mathcal{J}^{ij}\right)^\rho{}_\sigma\right)\alpha^\sigma(\vec{p})
\end{aligned}
\tag{28}
$$

其中记 $\tilde{J}^{\mu\nu}\equiv i(p^\mu\frac{\partial}{\partial p_\nu}-p^\nu\frac{\partial}{\partial p_\mu})$, 同时利用了 $\tilde{J}^{ij}\omega_{\vec{p}}=0$。

相似地，可以验证

$$\left[M^{ij}, E^{\rho}(x)\right] = J^{ij} E^{\rho}(x) + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma} E^{\sigma}(x) \tag{29}$$

则

$$\begin{aligned}
&\int d^3x e^{ik\cdot x}\left[M^{ij}, -\partial_0 A^{+\rho}(x)\right] \\
=&\int d^3x e^{ik\cdot x}\left[M^{ij}, E^{+\rho}(x)\right] \\
=&\int d^3x e^{ik\cdot x}\left(J^{ij}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\int \frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}(i\omega_{\vec{p}})\alpha^{\sigma}(\vec{p})e^{-ipx} \\
=&\int \frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}(i\omega_{\vec{p}})\alpha^{\sigma}(\vec{p})\left(\tilde{J}^{ji}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\int d^3x e^{i(k-p)x} \\
=&\int \frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}(i\omega_{\vec{p}})\alpha^{\sigma}(\vec{p})\left(-\tilde{J}^{ij}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})x^0} \\
=&\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})\left(\tilde{J}^{ij}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\alpha^{\sigma}(\vec{p})e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})x^0} \\
=&\frac{i}{2}\left(\tilde{J}^{ij}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\alpha^{\sigma}(\vec{p})
\end{aligned} \tag{30}$$

所以，

$$i\int d^3x e^{ik\cdot x}\left[M^{ij}, (\partial_0 - i\omega_{\vec{k}})A^{+\rho}(x)\right] = \left(\tilde{J}^{ij}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\alpha^{\sigma}(\vec{p}) \tag{31}$$

同理，

$$i\int d^3x e^{ik\cdot x}\left[M^{ij}, (\partial_0 - i\omega_{\vec{k}})A^{-\rho}(x)\right] = 0 \tag{32}$$

$$i\int d^3x e^{-ik\cdot x}\left[M^{ij}, (\partial_0 + i\omega_{\vec{k}})A^{-\rho}(x)\right] = \left(\tilde{J}^{ij}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\alpha^{*\sigma}(\vec{p}) \tag{33}$$

$$i\int d^3x e^{-ik\cdot x}\left[M^{ij}, (\partial_0 + i\omega_{\vec{k}})A^{+\rho}(x)\right] = 0 \tag{34}$$

进而，

$$\left[M^{ij}, a(\vec{k},\lambda)\right] = \epsilon_{\rho}(\vec{k},\lambda)\left(\tilde{J}^{ij}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\alpha^{\sigma}(\vec{p}) \tag{35}$$

$$\left[M^{ij}, a^{\dagger}(\vec{k},\lambda)\right] = \epsilon^{*}_{\rho}(\vec{k},\lambda)\left(\tilde{J}^{ij}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\alpha^{*\sigma}(\vec{p}) \tag{36}$$

即

$$\left[M^{ij}, a(\vec{k},\lambda)\right] = \epsilon_{\rho}(\vec{k},\lambda)\left(\tilde{J}^{ij}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\sum_{\lambda'}\epsilon^{*\sigma}(\vec{k},\lambda')a(\vec{k},\lambda') \tag{37}$$

$$\left[M^{ij}, a^{\dagger}(\vec{k},\lambda)\right] = \epsilon^{*}_{\rho}(\vec{k},\lambda)\left(\tilde{J}^{ij}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\sum_{\lambda'}\epsilon^{\sigma}(\vec{k},\lambda')a^{\dagger}(\vec{k},\lambda') \tag{38}$$

而对于

$$\begin{aligned}
[M^{0j}, a(\vec{k},\lambda)] =& i\int d^3x e^{ik\cdot x}\epsilon_{\rho}(\vec{k},\lambda)\left(J^{0j}\delta^{\rho}_{\sigma} + \left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)(\partial_0 - i\omega_{\vec{k}})A^{\sigma}(x) \\
&+ i\int d^3x e^{ik\cdot x}\epsilon_{\rho}(\vec{k},\lambda)(i\partial^j)A^{\rho}(x)
\end{aligned} \tag{39}$$

$$
\begin{aligned}
&\int d^3xe^{ik\cdot x}\left[M^{0j},A^{+\rho}(x)\right]\\
=&\int d^3xe^{ik\cdot x}\left(J^{0j}\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)A^{+\sigma}(x)\\
=&\int d^3xe^{ik\cdot x}\left(J^{0j}\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)\int\frac{d^3p}{(2\pi)^32\omega_{\vec{p}}}\alpha^\sigma(\vec{p})e^{-ipx}\\
=&\int\frac{d^3p}{(2\pi)^32\omega_{\vec{p}}}\alpha^\sigma(\vec{p})\int d^3x\left(i(-ix^0p^j+ix^j\omega_{\vec{p}})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)e^{i(k-p)x}\\
=&\int\frac{d^3p}{(2\pi)^32\omega_{\vec{p}}}\alpha^\sigma(\vec{p})e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})x^0}\left((x^0p^j-i\omega_{\vec{p}}\frac{\partial}{\partial p_j})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)\int d^3xe^{-i(\vec{k}-\vec{p})\cdot\vec{x}}\\
=&\int\frac{d^3p}{(2\pi)^32\omega_{\vec{p}}}\alpha^\sigma(\vec{p})e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})x^0}\left((x^0p^j-i\omega_{\vec{p}}\frac{\partial}{\partial p_j})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})\\
=&\int\frac{d^3p}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})\left((x^0p^j+i\frac{\partial}{\partial p_j}\omega_{\vec{p}})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)\frac{1}{2\omega_{\vec{p}}}\alpha^\sigma(\vec{p})e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})x^0}\\
=&\frac{1}{2\omega_{\vec{p}}}\left((2x^0k^j+i\omega_{\vec{k}}\frac{\partial^j\alpha^\sigma(\vec{k})}{\alpha^\sigma(\vec{k})})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)\alpha^\sigma(\vec{k})
\end{aligned}
\tag{40}
$$

与

$$
\begin{aligned}
&-\int d^3xe^{ik\cdot x}\left(J^{0j}\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)\partial_0A^{+\sigma}(x)\\
=&\int d^3xe^{ik\cdot x}\left(J^{0j}\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)E^{+\sigma}(x)\\
=&\int d^3xe^{ik\cdot x}\left(J^{0j}\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)\int\frac{d^3p}{(2\pi)^32\omega_{\vec{p}}}(i\omega_{\vec{p}})\alpha^\sigma(\vec{p})e^{-ipx}\\
=&\frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\alpha^\sigma(\vec{p})\int d^3x\left(i(-ix^0p^j+ix^j\omega_{\vec{p}})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)e^{i(k-p)x}\\
=&\frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\alpha^\sigma(\vec{p})e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})x^0}\left((x^0p^j-i\omega_{\vec{p}}\frac{\partial}{\partial p_j})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)\int d^3xe^{-i(\vec{k}-\vec{p})\cdot\vec{x}}\\
=&\frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\alpha^\sigma(\vec{p})e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})x^0}\left((x^0p^j-i\omega_{\vec{p}}\frac{\partial}{\partial p_j})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})\\
=&\frac{i}{2}\int\frac{d^3p}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})\left((x^0p^j+i\frac{\partial}{\partial p_j}\omega_{\vec{p}})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)\alpha^\sigma(\vec{p})e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})x^0}\\
=&\frac{i}{2}\left((2x^0k^j+\frac{ip^j}{\omega_{\vec{k}}}+i\omega_{\vec{k}}\frac{\partial^j\alpha^\sigma(\vec{k})}{\alpha^\sigma(\vec{k})})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)\alpha^\sigma(\vec{k})
\end{aligned}
\tag{41}
$$

那么，

$$
i\int d^3xe^{ik\cdot x}\left(J^{0j}\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)(\partial_0-i\omega_{\vec{k}})A^{+\sigma(x)}=\left((2x^0k^j+\frac{ik^j}{2\omega_{\vec{k}}}+i\omega_{\vec{k}}\frac{\partial}{\partial k_j})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)\alpha^\sigma(\vec{k})
\tag{42}
$$

于是，

$$
i\int d^3xe^{ik\cdot x}\left[M^{0j},(\partial_0-i\omega_{\vec{k}})A^{+\rho}(x)\right]=\left((2x^0k^j+\frac{ip^j}{\omega_{\vec{k}}}+i\omega_{\vec{k}}\frac{\partial}{\partial k_j})\delta^\rho_\sigma+\left(\mathcal{J}^{0j}\right)^\rho{}_\sigma\right)\alpha^\sigma(\vec{k})
\tag{43}
$$

第二项中

$$
\begin{aligned}
&\int d^3xe^{ik\cdot x}\left[M^{0j},A^{-\rho}(x)\right]\\
=&\int d^3xe^{ik\cdot x}\left(J^{0j}A^{-\rho}(x)+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}A^{-\sigma}(x)\right)\\
=&\int d^3xe^{ik\cdot x}\left(J^{0j}\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)\int\frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}\alpha^{*\sigma(\vec{p})}e^{ipx}\\
=&\int\frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}\alpha^{*\sigma(\vec{p})}\int d^3x\left(i(ix^0p^j-ix^j\omega_{\vec{p}})\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)e^{i(k+p)x}\\
=&\int\frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}\alpha^{\sigma}(\vec{p})e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})x^0}\left((-x^0p^j+i\omega_{\vec{p}}\frac{\partial}{\partial p_j})\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)\int d^3xe^{-i(\vec{k}+\vec{p})\cdot\vec{x}}\\
=&\int\frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}\alpha^{\sigma}(\vec{p})e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})x^0}\left((-x^0p^j+i\omega_{\vec{p}}\frac{\partial}{\partial p_j})\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)(2\pi)^3\delta^{(3)}(\vec{p}+\vec{k})\\
=&\int\frac{d^3p}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\vec{p}+\vec{k})\left((-x^0p^j-i\frac{\partial}{\partial p_j}\omega_{\vec{p}})\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)\frac{1}{2\omega_{\vec{p}}}\alpha^{\sigma}(\vec{p})e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})x^0}\\
=&\frac{1}{2\omega_{\vec{k}}}\left((-i\omega_{\vec{k}}\frac{\partial^j\alpha^{\sigma}(-\vec{k})}{\alpha^{\sigma}(-\vec{k})})\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)\alpha^{\sigma}(-\vec{p})e^{i2\omega_{\vec{k}}x^0}
\end{aligned}
\tag{44}
$$

$$
-\int d^3xe^{ik\cdot x}\left(J^{0j}\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)\partial_0A^{-\sigma}(x)=-\frac{i}{2}\left((-\frac{ik^j}{\omega_{\vec{k}}}-i\omega_{\vec{k}}\frac{\partial^j\alpha^{\sigma}(-\vec{k})}{\alpha^{\sigma}(-\vec{k})})\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)\alpha^{\sigma}(-\vec{k})e^{i2\omega_{\vec{k}}x^0}
\tag{45}
$$

那么，

$$
i\int d^3xe^{ik\cdot x}\left(J^{0j}\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)(\partial_0-i\omega_{\vec{k}})A^{-\sigma}(x)=\left(-\frac{ik^j}{2\omega_{\vec{k}}}\right)\alpha^{\sigma}(-\vec{k})e^{i2\omega_{\vec{k}}x^0}
\tag{46}
$$

于是，

$$
i\int d^3xe^{ik\cdot x}\left[M^{0j},(\partial_0-i\omega_{\vec{k}})A^{-\rho}(x)\right]=0
\tag{47}
$$

所以，

$$
[M^{0j},a(\vec{k},\lambda)]=\epsilon_{\rho}(\vec{k},\lambda)\left((2x^0k^j+\frac{ik^j}{\omega_{\vec{k}}}+i\omega_{\vec{k}}\frac{\partial}{\partial k_j})\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)\alpha^{\sigma}(\vec{k})
\tag{48}
$$

同理，

$$
[M^{0j},a^{\dagger}(\vec{k},\lambda)]=\epsilon^{*}_{\rho}(\vec{k},\lambda)\left(-(2x^0k^j+\frac{ik^j}{\omega_{\vec{k}}}+i\omega_{\vec{k}}\frac{\partial}{\partial k_j})\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)\alpha^{*\sigma}(\vec{k})
\tag{49}
$$

综上所述，

$$
\left[M^{ij},a(\vec{k},\lambda)\right]=\epsilon_{\rho}(\vec{k},\lambda)\left(\tilde{J}^{ij}\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\sum_{\lambda'}\epsilon^{*\sigma}(\vec{k},\lambda')a(\vec{k},\lambda')
\tag{50}
$$

$$
\left[M^{ij},a^{\dagger}(\vec{k},\lambda)\right]=\epsilon^{*}_{\rho}(\vec{k},\lambda)\left(\tilde{J}^{ij}\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{ij}\right)^{\rho}{}_{\sigma}\right)\sum_{\lambda'}\epsilon^{\sigma}(\vec{k},\lambda')a^{\dagger}(\vec{k},\lambda')
\tag{51}
$$

$$
[M^{0j},a(\vec{k},\lambda)]=\epsilon_{\rho}(\vec{k},\lambda)\left((2x^0k^j+\frac{ik^j}{\omega_{\vec{k}}}+i\omega_{\vec{k}}\frac{\partial}{\partial k_j})\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)\sum_{\lambda'}\epsilon^{*\sigma}(\vec{k},\lambda')a(\vec{k},\lambda')
\tag{52}
$$

$$
[M^{0j},a^{\dagger}(\vec{k},\lambda)]=\epsilon^{*}_{\rho}(\vec{k},\lambda)\left(-(2x^0k^j+\frac{ik^j}{\omega_{\vec{k}}}+i\omega_{\vec{k}}\frac{\partial}{\partial k_j})\delta^{\rho}_{\sigma}+\left(\mathcal{J}^{0j}\right)^{\rho}{}_{\sigma}\right)\sum_{\lambda'}\epsilon^{\sigma}(\vec{k},\lambda')a^{\dagger}(\vec{k},\lambda')
\tag{53}
$$

习题 4

接上题, 设 $\vec{k}=(0,0,1)$ 。定义角动量算符 $J^i=\frac{1}{2}\epsilon^{ijk}M_{jk}$, 其中 ϵ^{ijk} 是三维空间的完全反对称张量, $\epsilon^{123}=1$ 。求

$$\left[J^3, a(\vec{k},\lambda)\right] \text{ 和 } \left[J^3, a^\dagger(\vec{k},\lambda)\right] \tag{54}$$

解:

$$\left[J^3, a(\vec{k},\lambda)\right] = \left[M_{12}, a(\vec{k},\lambda)\right] = \epsilon_\rho(\vec{k},\lambda)\left(\tilde{J}_{12}\delta^\rho_\sigma + (\mathcal{J}_{12})^\rho{}_\sigma\right)\sum_{\lambda'}\epsilon^{*\sigma}(\vec{k},\lambda')a(\vec{k},\lambda') \tag{55}$$

而由于 $\vec{k}=(0,0,1)$,

$$\begin{aligned}\left[J^3, a(\vec{k},\lambda)\right] &= \epsilon_\rho(\vec{k},\lambda)\,(\mathcal{J}_{12})^\rho{}_\sigma\sum_{\lambda'}\epsilon^{*\sigma}(\vec{k},\lambda')a(\vec{k},\lambda')\\ &= i\epsilon^1(\vec{k},\lambda)\sum_{\lambda'}\epsilon^{*2}(\vec{k},\lambda')a(\vec{k},\lambda') - i\epsilon^2(\vec{k},\lambda)\sum_{\lambda'}\epsilon^{*1}(\vec{k},\lambda')a(\vec{k},\lambda')\end{aligned} \tag{56}$$

$$\begin{aligned}\left[J^3, a^\dagger(\vec{k},\lambda)\right] &= \epsilon^*_\rho(\vec{k},\lambda)\,(\mathcal{J}_{12})^\rho{}_\sigma\sum_{\lambda'}\epsilon^{\sigma}(\vec{k},\lambda')a(\vec{k},\lambda')\\ &= i\epsilon^{1*}(\vec{k},\lambda)\sum_{\lambda'}\epsilon^{2}(\vec{k},\lambda')a^\dagger(\vec{k},\lambda') - i\epsilon^{2*}(\vec{k},\lambda)\sum_{\lambda'}\epsilon^{1}(\vec{k},\lambda')a^\dagger(\vec{k},\lambda')\end{aligned} \tag{57}$$

量子场论的洛伦兹规范要求 $\sum_\lambda a(\vec{p},\lambda)k^\mu\epsilon_\mu(\vec{k},\lambda)|\Psi\rangle_{phy.}=0$, 不妨取 $\epsilon^\mu(\vec{k},\lambda)=\delta^\mu_\lambda$, 那么

$$\left[J^3, a(\vec{k},\lambda)\right] = i\delta^1_\lambda a(\vec{k},2) - i\delta^2_\lambda a(\vec{k},1) \tag{58}$$

和

$$\left[J^3, a^\dagger(\vec{k},\lambda)\right] = i\delta^1_\lambda a^\dagger(\vec{k},2) - i\delta^2_\lambda a^\dagger(\vec{k},1) \tag{59}$$

习题 5

接上题, 定义 $a_\pm(\vec{k}) = \pm\frac{1}{\sqrt{2}}[a(\vec{k},1)\mp ia(\vec{k},2)]$, $a^\dagger_\pm(\vec{k}) = \pm\frac{1}{\sqrt{2}}\left[a^\dagger(\vec{k},1)\pm ia^\dagger(\vec{k},2)\right]$ 。证明 $a^\dagger_\pm(\vec{k})$ 产生一个具有确定 z 方向角动量的态, 并求其螺旋度。

解:

$$\left[J^3, a_\pm(\vec{k},\lambda)\right] = -a_\pm(\vec{k},\lambda) \tag{60}$$

$$\left[J^3, a^\dagger_\pm(\vec{k},\lambda)\right] = +a^\dagger_\pm(\vec{k},\lambda) \tag{61}$$

因此

$$\left[J^3, a_\pm(\vec{k},\lambda)\right]|0\rangle = -a_\pm(\vec{k},\lambda)|0\rangle = J^3a_\pm(\vec{k},\lambda)|0\rangle \tag{62}$$

$$\left[J^3, a^\dagger_\pm(\vec{k},\lambda)\right] = +a^\dagger_\pm(\vec{k},\lambda) = J^3a^\dagger_\pm(\vec{k},\lambda)|0\rangle \tag{63}$$

所以 $a^\dagger_\pm(\vec{k})$ 产生一个具有确定 z 方向角动量为 $+1$ 的态, 其螺旋度 $h=\frac{J\cdot k}{k}=J^3=1$。

习题 6

通过诺特定理推导无质量矢量场的能量动量张量 $T_{\mu\nu}$ 。请判断你所求出的能动张量是否是对称张量？是否是规范不变的？

解：

时空平移变化下 $\delta^\mu A^\nu = \partial^\mu A^\nu$，

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\lambda\right)} \partial^\nu A_\lambda - \eta^{\mu\nu}\mathcal{L} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}\eta^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa} \tag{64}$$

非对称张量，非规范不变。实际上 $T^{\mu\nu} + \partial_\lambda(F^{\mu\lambda}A^\nu)$ 才是对称且规范不变的量。

习题 7

计算高能极限下 $e^+e^- \to \gamma\gamma$ 的树图散射振幅和微分散射截面 $\frac{d\sigma}{d\cos\theta}$，其中 θ 是出射光子相对入射方向的夹角。在高能极限下，电子质量可设为零。

解：

$$i\mathcal{M}(e^-(1)e^+(2) \to \gamma(3)\gamma(4)) = -e^2 \varepsilon_3^\mu \varepsilon_4^\nu \bar{v}_2 \left[\gamma_\nu \left(\frac{\not{p}_1 - \not{k}_3}{t}\right)\gamma_\mu + \gamma_\mu\left(\frac{\not{p}_1 - \not{k}_4}{u}\right)\gamma_\nu\right] u_1 \tag{65}$$

对于

$$\begin{aligned} i\mathcal{M}_{-+\lambda_3\lambda_4} = &- e^2 \left\langle 2 \left| \not{\epsilon}_{\lambda_4}\left(k_4; q_4\right)\left(\not{p}_1 - \not{k}_3\right)\not{\epsilon}_{\lambda_3}\left(k_3; q_3\right)\right| 1\right] / t \\ &- e^2 \left\langle 2 \left| \not{\epsilon}_{\lambda_3}\left(k_3; q_3\right)\left(\not{p}_1 - \not{k}_4\right)\not{\epsilon}_{\lambda_4}\left(k_4; q_4\right)\right| 1\right] / u. \end{aligned} \tag{66}$$

通过恰当选择 q_3, q_4，可以发现仅当 λ_3 与 λ_4 不同时候，不为 0. 而

$$i\mathcal{M}_{-++-} = 2e^2 \frac{\langle 24\rangle^2}{\langle 13\rangle\langle 23\rangle} \tag{67}$$

$$i\mathcal{M}_{-+-+} = 2e^2 \frac{\langle 23\rangle^2}{\langle 14\rangle\langle 24\rangle} \tag{68}$$

所以

$$\frac{1}{4}\sum |\mathcal{M}|^2 = 2e^4\left(\frac{t}{u} + \frac{u}{t}\right) \tag{69}$$

而由

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\frac{p_f}{p_i}\left|\mathcal{M}_{fi}\right|^2 \tag{70}$$

得到

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{2\pi\alpha^2}{s}\frac{1+\cos^2\theta}{\sin^2\theta} \tag{71}$$